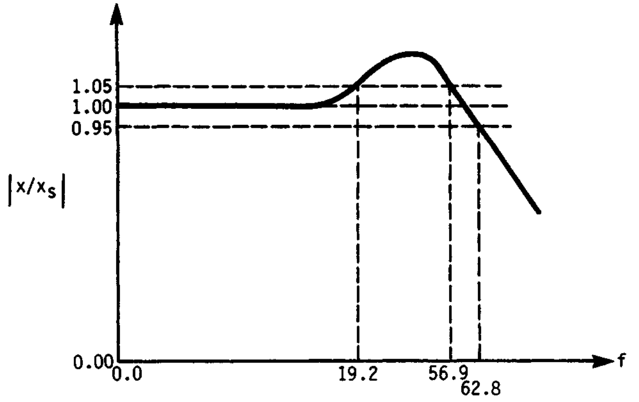


FIGURE 3.23 Frequency response of strip-chart recorder of Example 4.

Input frequency ω, rad/s	Amplitude, V Input	Amplitude, V Output	Phase angle (lag), °
0	10.0	10.0	0
5	10.0	10.0	10
10	10.0	10.2	20
15	10.0	10.6	30
20	10.0	11.0	45
25	10.0	12.2	90

It follows that

$$e_o = 10\left(\frac{10}{10}\right) + 5.8\left(\frac{10.0}{10.0}\right)\cos\left(5t - \frac{10\pi}{180}\right) + 3.2\left(\frac{10.2}{10.0}\right)\cos\left(10t - \frac{20\pi}{180}\right)$$

$$+ 1.8\left(\frac{11.0}{10.0}\right)\cos\left(20t - \frac{45\pi}{180}\right)$$

$$= 10 + 5.8\cos(5t - 0.174) + 3.26\cos(10t - 0.349) + 1.98\cos(20t - 0.785)$$

3.7 SELECTED MEASURING-SYSTEM COMPONENTS AND EXAMPLES

3.7.1 Operational Amplifiers

Operational amplifiers [3.8] used in measuring systems have the basic configuration shown in Fig. 3.24. The *operational amplifier* is composed of a high-gain voltage amplifier coupled with both input and feedback impedances. The characteristics of

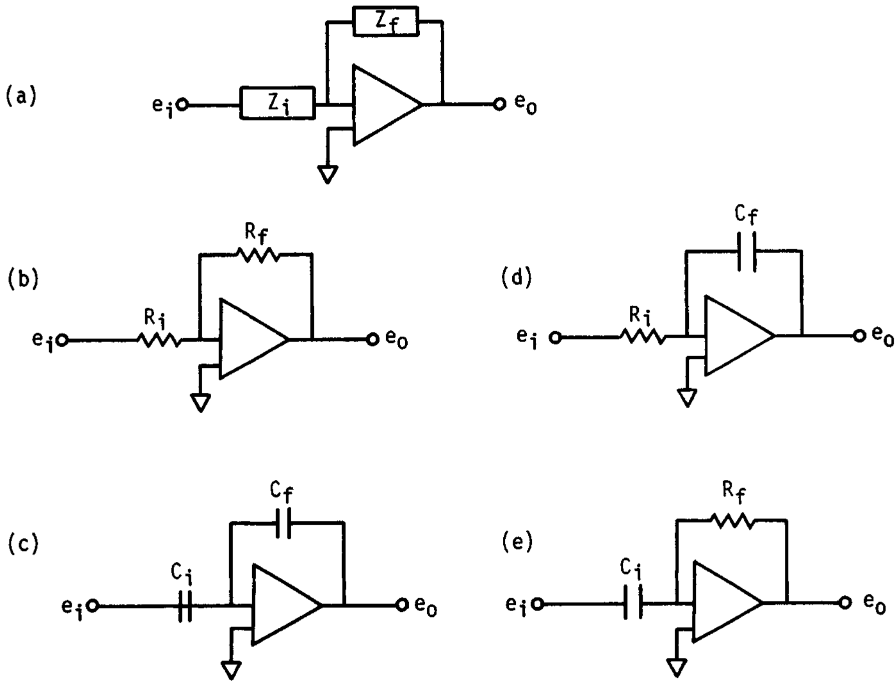


FIGURE 3.24 Operational amplifier circuit. (*a*) General; (*b*) voltage amplifier; (*c*) charge amplifier; (*d*) integrator; (*e*) differentiator.

the operational amplifier depend on the feedback impedance Z_f and input impedance Z_i, selected according to Eq. (3.20):

$$\frac{e_o}{e_i} = -\frac{Z_f}{Z_i} \tag{3.20}$$

The relations between input and output voltage for the specific configurations shown in Fig. 3.24 are as follows:
Voltage amplifier:

$$\frac{e_o}{e_i} = -\frac{R_f}{R_i} \tag{3.21}$$

Charge amplifier:

$$\frac{e_o}{e_i} = -\frac{C_i}{C_f} \tag{3.22}$$

Integrator:

$$e_o = -\frac{1}{R_i C_f} \int_0^t e_i \, dt + e_o(0) \tag{3.23}$$

Differentiator:

$$e_o = -R_f C_i \frac{de_i}{dt} \tag{3.24}$$

3.7.2 Piezoelectric Crystal

Piezoelectric crystals [3.9] are specific crystals of such materials as quartz, barium titinate, and lead zirconate which, when properly heated and quenched, demonstrate the piezoelectric phenomenon. The *piezoelectric phenomenon* is that the crystal, when stressed, produces an electric charge on its surfaces. If the crystal is a wafer of thickness t and its surfaces are coated with (or touching) conductive plates, the plates become a capacitor of plate area A, spacing t, and dielectric property ε of the piezoelectric material. The voltage developed from the piezoelectric crystal from any input (force, pressure, acceleration, stress, etc.) is

$$e_o = S_e x \tag{3.25}$$

where S_e = voltage sensitivity and x = input variable. The voltage sensitivity depends on the fundamental charge sensitivity of the piezoelectric crystal:

$$S_e = \frac{S_q}{C_c} \tag{3.26}$$

where $S_q = q/x$ and C_c = crystal capacitance, given by

$$C_c = \frac{KA\varepsilon}{t} \tag{3.27}$$

K is a constant which depends on the geometry and the units of the parameters in the preceding equation.

When the piezoelectric crystal is coupled via lead wires with capacitance, the voltage sensitivity and output voltage are reduced according to the relation

$$e_o = S_e x = \frac{S_q}{C_T} x \tag{3.28}$$

where C_T = total capacitance of the combination of piezoelectric crystal, lead wires, and readout device and is equal to

$$C_T = C_c + C_{lw} + C_{rd} \tag{3.29}$$

The equivalent circuits of the piezoelectric crystal are given in Fig. 3.25. The piezoelectric crystal has a dynamic response that is approximately that of an undamped second-order system. The circuit components of the piezoelectric crystal have a dynamic response that is approximately that of a first-order system. The typical frequency response of the piezoelectric transducer is that shown in Fig. 3.26 and is the combination of the crystal and circuit responses.

When the piezoelectric crystal is coupled with a voltage amplifier, the output voltage of the measuring system is dependent on lead-wire capacitance according to the relation

$$e_o = -\frac{R_f}{R_i} S_e x = -\frac{R_f}{R_i} \frac{S_q}{C_T} x \tag{3.30}$$

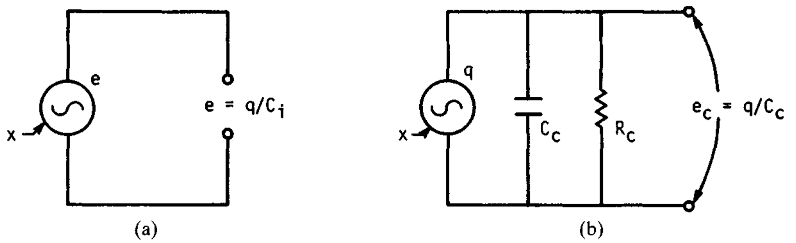


FIGURE 3.25 Equivalent circuits of a piezoelectric crystal. (*a*) Voltage generator equivalent circuit; (*b*) charge generator equivalent circuit.

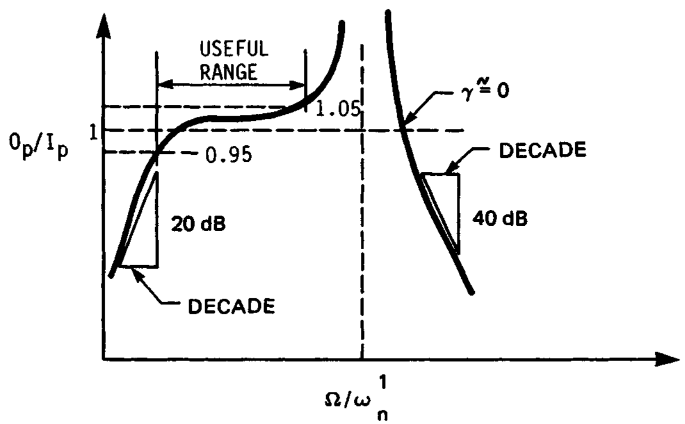


FIGURE 3.26 Composite frequency response of a piezoelectric transducer.

where $C_T = C_c + C_{lw} + C_a$ and R_f/R_i = the ratio of feedback to input resistance on the operational amplifier used for voltage amplification. Thus long lead wires or high lead-wire capacitance will significantly decrease the output voltage of the measuring system when using a voltage amplifier. The use of a charge amplifier avoids the problem of capacitance of the input lead wires, as shown by the relation

$$e_o = -\frac{C_i}{C_f} S_e x = -\frac{C_i}{C_f} \frac{S_q}{C_T} x \tag{3.31}$$

where C_i equals C_T and is the total capacitance at the input to the charge amplifier. Thus with a charge amplifier the voltage sensitivity S_e of the system depends only on the basic crystal charge sensitivity S_q and the charge amplifier feedback capacitance C_f and not on the input capacitance.

Example 5. A piezoelectric accelerometer is to be used to measure the vibration of an automotive engine as installed in a particular test cell. The pertinent characteristics of the transducer, cable, charge amplifier, and cathode-ray oscilloscope used in the acceleration measuring system are given in the following table:

Characteristic	Piezoelectric accelerometer	Cable	Charge amplifier	Cathode-ray oscilloscope	Voltage amplifier
Charge sensitivity, pC/g	123				
Natural frequency, Hz	30 000				
Capacitance, pF	8 600	300			
Resistance, Ω	10^{12}	Negligible			
Feedback capacitance, pF	—	—	10^3		
Input resistance, Ω	—	—	10^8	10^6	
Input capacitance, pF	—	—	50	50	
$\frac{R_f}{R_i}$	—	—	—	—	0.123

The circuit diagram is shown in Fig. 3.27.

Determine the sensitivity of the *measuring system* if the "charge sensitivity" setting on the charge amplifier is adjusted to a value of 0.123.

The following equation gives the sensitivity:

$$\frac{R_t}{R_T} = 0.123 \qquad \frac{e_o}{x} = \frac{R_T}{R_t}\frac{S_q}{C_f} = \frac{123}{0.123}\frac{1}{10^3} = 1\text{ V/g}$$

What sensitivity setting should be selected on the cathode-ray oscilloscope?

For 1 g/cm, use 1 V/cm.

For 0.1 g/cm, use 0.1 V/cm.

What range of acceleration can be measured if a maximum output of 10 V is available?

Range of acceleration is found as follows:

$$x_{\max} = \frac{e_o|_{\max}}{S} = \frac{10}{1\text{ V/g}} = 10.0\text{ g}$$

What is the voltage sensitivity of the accelerometer?

1. If the charge amplifier is used,

$$\frac{e_i}{x} = \frac{q/x}{C_i} = \frac{S_q}{C_i} \qquad \frac{e_o}{x} = -\frac{e_i}{x}\frac{C_i}{C_f} = -\frac{S_q}{C_f}$$

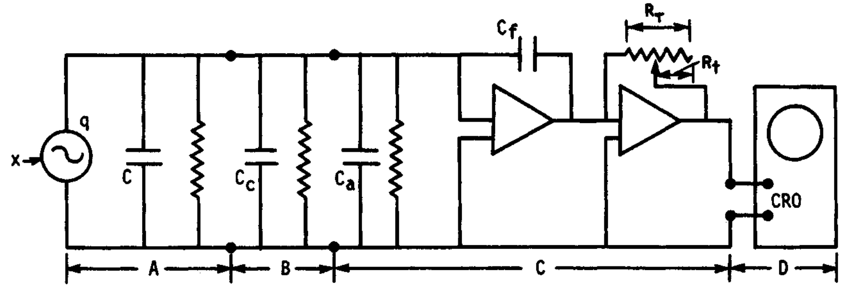


FIGURE 3.27 Circuit diagram of the vibration-measuring system. A, piezoelectric crystal; B, cable; C, charge amplifier coupled to a voltage amplifier; D, cathode-ray oscilloscope.

Thus $$\text{Voltage sensitivity} = \frac{123 \text{ pC/g}}{10^3 \text{ pF}} = 123 \text{ mV/g}$$

2. If the accelerometer is connected directly to the cathode-ray oscilloscope (no charge amplifier),

$$\frac{e_o}{e_i} = -G \qquad \frac{e_o}{x} = -G\frac{e_i}{x} = -G\frac{S_q}{C_i} = -\frac{G(123) \text{ pC/g}}{(8600 + 300 + 50) \text{ pF}}$$

Thus $$\text{Voltage sensitivity} = -13.7G \text{ mV/g}$$

If the accelerometer has zero damping, what would be the largest frequency of input vibration allowable to have no more than a 1 percent error? If the engine has eight cylinders and operates at 4000 rpm, will the measuring system work?
The computations are as follows:

$$\frac{8(4000)}{60} = 533.3 \text{ Hz} = \text{vibration frequency expected}$$

$$\text{AF}_{\text{HF}} = \frac{1}{1 - (\Omega/\omega n)^2} = 1.01 \qquad \Omega/\omega_n = 0.0995 \qquad \Omega_{\text{H}} = 2985 \text{ Hz}$$

$$\text{AF}_{\text{LF}} = \frac{1}{\sqrt{1 + (\tau\Omega)^2}} = 0.99$$

$$\tau = R_T C_T = 10^6(8650 \times 10^{-12}) = 0.00865 \text{ s}$$

$$\tau\Omega = 0.1425 \qquad \Omega_{\text{L}} = 15.83 \text{ rad/s} = 2.5 \text{ Hz}$$

The frequency response of the vibration measurement system is satisfactory and is shown in Fig. 3.28.

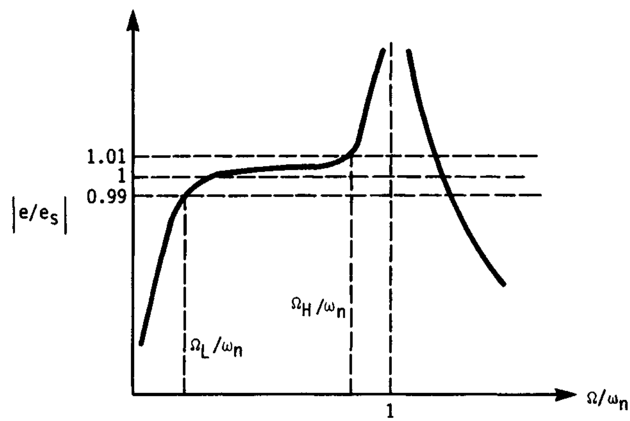


FIGURE 3.28 Frequency response of the vibration-measuring system.

3.7.3 Ballast-Type Circuit

A basic circuit used in measurement applications is the ballast-type circuit shown in Fig. 3.29. The relation between input and output voltage is given by

$$\frac{e_o}{e_i} = \frac{Z_L}{Z_B + Z_L} \tag{3.32}$$

where Z_L = load impedance and Z_B = ballast impedance.

When Z_L and Z_B are capacitance C and resistance R, respectively, the circuit is used as a low-pass filter with output voltage and phase shift given by Eqs. (3.33) and (3.34), respectively, where ω is the frequency of the input signal:

$$\left|\frac{e_o}{e_i}\right| = \sqrt{\frac{1}{1 + (RC\omega)^2}} \tag{3.33}$$

$$\phi = \tan^{-1}(RC\omega) \tag{3.34}$$

When Z_L and Z_B are resistance and capacitance, respectively, the circuit is used as a high-pass filter.

The output voltage and phase shift are then given by Eqs. (3.35) and (3.36), respectively:

$$\left|\frac{e_o}{e_i}\right| = \sqrt{\frac{(RC\omega)^2}{1 + (RC\omega)^2}} \tag{3.35}$$

$$\phi = \tan^{-1}\left(\frac{1}{RC\omega}\right) \tag{3.36}$$

An example of this type of circuit is the ac coupling circuit at the input of a cathode-ray oscilloscope. When Z_L is that of an impedance-based detector transducer such as a resistance thermometer or strain gauge, the voltage e_i is that of the auxiliary energy source and Z_B is an impedance used to limit the current flow to the detector transducer. If Joule (I^2R) heating would affect the transducer measurement, such as in resistance-thermometer or strain-gauge applications, the ability to limit current is important.

Example 6. The circuit of Fig. 3.30*a* is used as a coupling circuit between a detector transducer and a readout device. Determine and sketch the amplitude and phase characteristics of the coupling circuit (see Fig. 3.30*b*, *c*, and *d*). Determine the load-

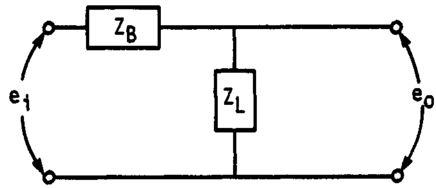


FIGURE 3.29 The ballast-type circuit.

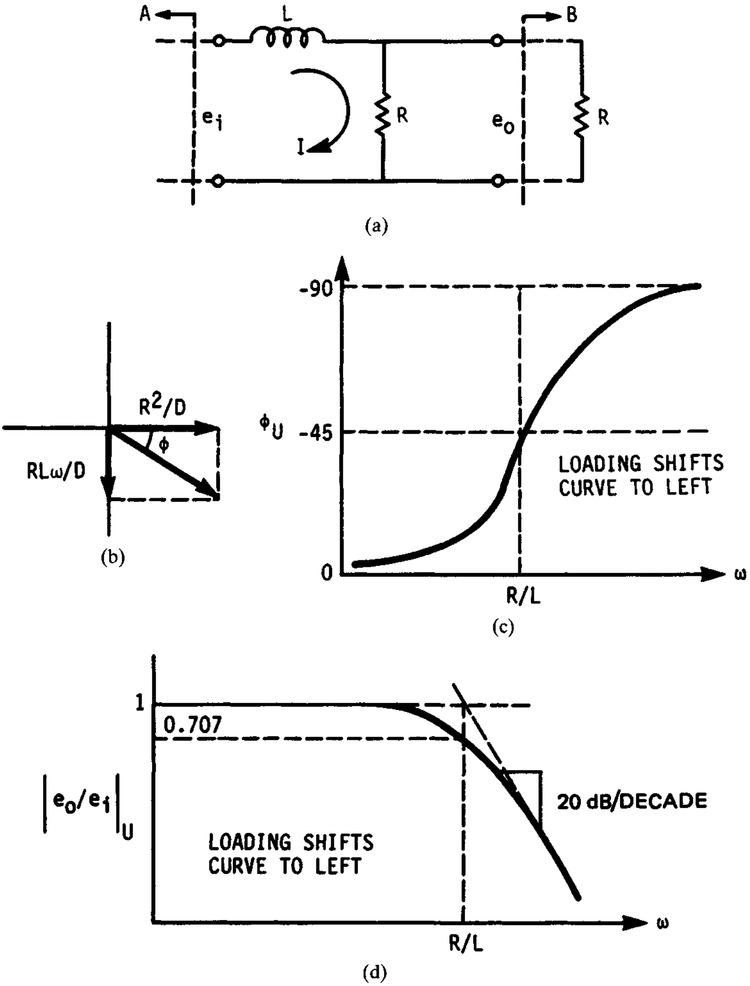


FIGURE 3.30 Coupling circuit example. A, detector transducer; B, readout. (*a*) Inductor and resistance in a ballast-type circuit; (*b*) real and complex components; (*c*) phase-shift characteristic; (*d*) frequency-response characteristic.

ing error if a readout device having an input impedance equal to R is connected to the circuit.

The equations are as follows:

$$e_o = IR$$

$$e_i = I(Z_L + R)$$

$$\left.\frac{e_o}{e_i}\right|_U = \left(\frac{R}{j\omega L + R}\right)\left(\frac{-j\omega L + R}{-j\omega L + R}\right) = \frac{R^2 - jRL\omega}{R^2 + (\omega L)^2} = \frac{R^2}{D} - \frac{jRL\omega}{D}$$

$$\left|\frac{e_o}{e_i}\right|_U = \sqrt{\left(\frac{R^2}{D}\right)^2 + \left(\frac{RL\omega}{D}\right)^2}$$

$$= \sqrt{\frac{R^4 + R^2L^2\omega^2}{R^4 + 2R^2\omega^2L^2 + (\omega L)^4}} = \sqrt{\frac{R^2(R^2 + L^2\omega^2)}{[R^2 + (\omega L)^2]^2}}$$

$$= \sqrt{\frac{R^2}{R^2 + (\omega L)^2}} = \sqrt{\frac{1}{1 + (\omega L/R)^2}}$$

$$\left.\frac{e_o}{e_i}\right|_L = \frac{R_{\text{eq}}^2}{D} - \frac{jR_{\text{eq}}L\omega}{D}$$

and

$$\left|\frac{e_o}{e_i}\right|_L = \sqrt{\frac{1}{1 + (\omega L/R_{\text{eq}})^2}} = \sqrt{\frac{1}{1 + (2\omega L/R)^2}}$$

$$LE \equiv \frac{|e_o/e_i|_U - |e_o/e_i|_L}{|e_o/e_i|_U} = 1 - \frac{e_o|_L}{e_o|_U} = 1 - I$$

$$= 1 - \sqrt{\frac{1 + (\omega L/R)^2}{1 + 4(\omega L/R)^2}}$$

$$\phi_L = \tan^{-1}\frac{2\omega L}{R} \qquad \phi_U = \tan^{-1}\frac{L\omega}{R}$$

Tables 3.2 and 3.3 give several examples of both ballast and bridge circuits used in instrumentation systems.

3.7.4 Bridge Circuit

The bridge circuit used in measurement circuits is shown in Fig. 3.31. For voltage excitation, e_i, the output Δe_o corresponds to the change in output voltage due to the change in the arm impedances of the bridge. The relationship between output voltage and impedance change in one arm of the bridge is given as follows:

$$\frac{e_o + \Delta e_o}{e_i} = \frac{(Z_1 + \Delta Z_1)Z_4 - Z_2Z_3}{(Z_1 + \Delta Z_1 + Z_2)(Z_3 + Z_4)} \tag{3.37}$$

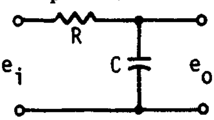
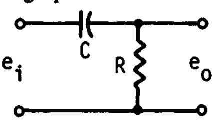
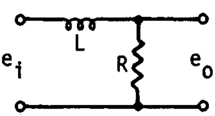
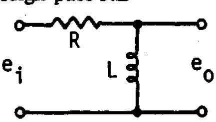
If initially the bridge is said to be "balanced," the output voltage e_o is zero and the relationship for the impedances in the bridge is given by the balance equation

$$\frac{Z_1}{Z_2} = \frac{Z_3}{Z_4} \tag{3.38}$$

This relation is used to measure an unknown impedance connected in a bridge circuit with three other impedances which are known. The reader is referred to Prensky [3.10] and to Table 3.3 for further information in this regard.

The ability of the bridge circuit to "zero" the output at any level of input transducer impedance allows the circuit to be used for the "balance" type of measurement, which is more accurate than the "unbalance" type of measurement commonly employed when using the ballast-type circuit.

TABLE 3.2 Typical Ballast-Type Circuits Used in Instrumentation Circuits

<table>
<tr><th>Ballast-type circuits</th><th>Magnitude response and phase shift</th></tr>
<tr><td>(a) Low-pass RC

</td><td>$\left|\frac{e_o}{e_i}\right| = \frac{1}{\sqrt{1+\omega^2\tau^2}} \approx \frac{1}{\omega\tau}$
$\phi = -\tan^{-1} R\omega C$</td></tr>
<tr><td>(b) High-pass RC

</td><td>$\left|\frac{e_o}{e_i}\right| = \frac{1}{\sqrt{1+1/\omega^2\tau^2}} \approx \omega\tau$
$\phi = \tan^{-1} \frac{1}{R\omega C}$</td></tr>
<tr><td>(c) Low-pass RL

</td><td>$\left|\frac{e_o}{e_i}\right| = \frac{1}{\sqrt{1+\omega^2\tau^2}} \approx \frac{1}{\omega\tau}$
$\phi = -\tan^{-1} \frac{\omega L}{R}$</td></tr>
<tr><td>(d) High-pass RL

</td><td>$\left|\frac{e_o}{e_i}\right| = \frac{1}{\sqrt{1+1/\omega^2\tau^2}} \approx \omega\tau$
$\phi = \tan^{-1} \frac{R}{\omega L}$</td></tr>
</table>

When all impedances are initially equal, the bridge is balanced, and the impedance change from an input signal is small compared to the original impedance, the bridge output voltage is linearized to

$$\Delta e_o \approx \frac{\Delta Z}{4Z} e_i \tag{3.39}$$

This equation can be used to predict the output of impedance-based transducers such as variable capacitors, variable inductances, or variable resistances (such as resistance thermometers or strain gauges) used in voltage-sensitive bridge circuits.

3.7.5 Strain Gauges

The strain gauge is a resistance R (usually in the form of a grid) wire or foil that changes when strained according to the relation

$$R = \rho \frac{L}{A} \tag{3.40}$$

TABLE 3.3 Typical Bridge Circuits Used in Instrumentation Circuits

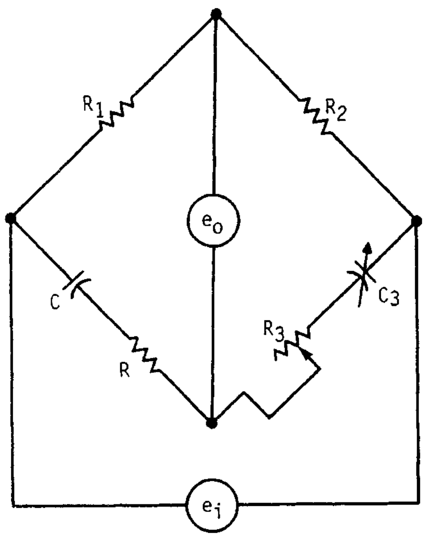
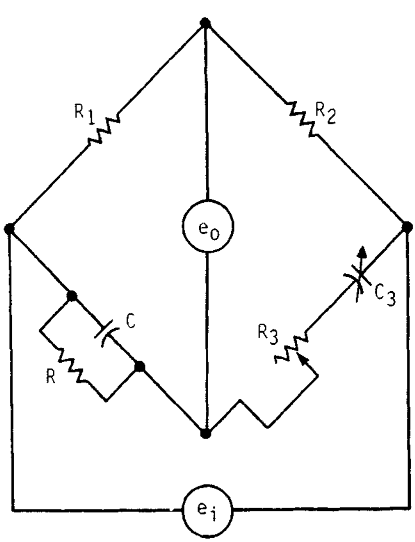
Bridge circuits	Balance relations
ac Wheatstone bridge 	$C = \frac{C_3 R_2}{R_1}$ $R = \frac{R_3 R_1}{R_2}$
Wein bridge 	$\frac{C}{C_3} = \frac{R_2}{R_1} - \frac{R_3}{R}$ $CC_3 = \frac{1}{\omega^2 R_3 R}$ If $C_3 = C$ and $R_3 = R$, $f = \frac{1}{2\pi R_3 C_3}$

TABLE 3.3 Typical Bridge Circuits Used in Instrumentation Circuits (*Continued*)

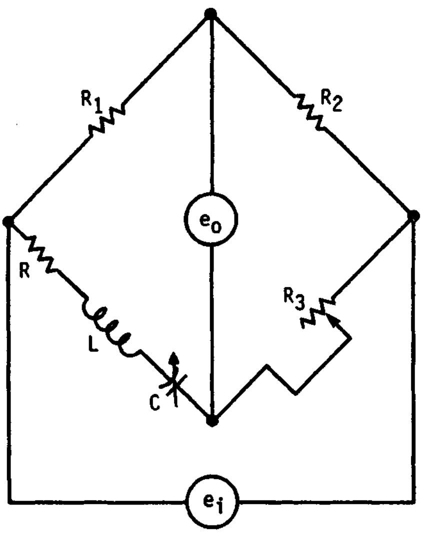
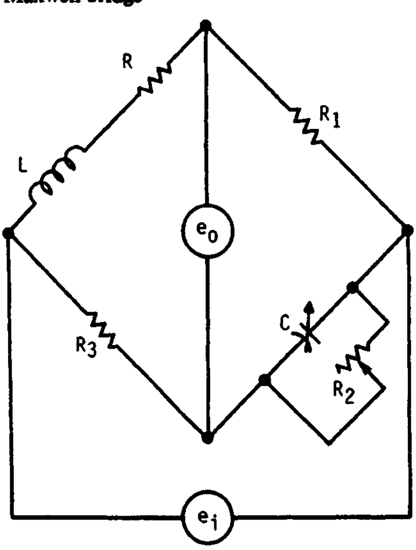
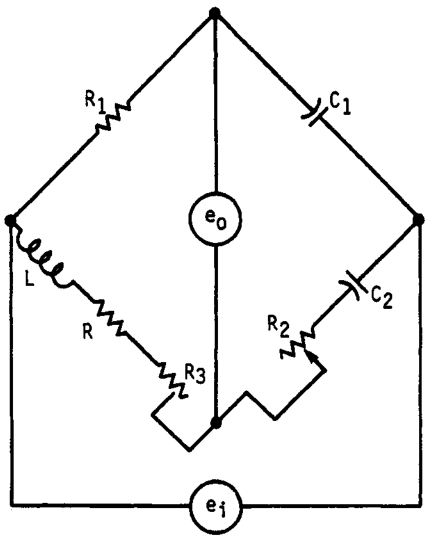
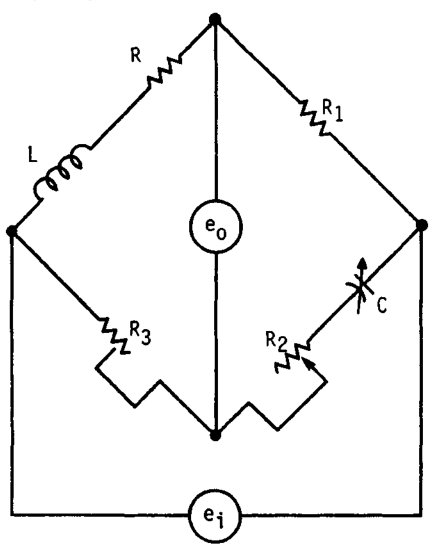
Bridge circuits	Balance relations
Resonance bridge  	$\omega^2 LC = 1$ $R = \dfrac{R_3 R_1}{R_2}$ At balance $f = \dfrac{1}{2\pi\sqrt{LC}}$
Maxwell bridge  	$L = R_1 R_3 C$ $R = \dfrac{R_1 R_3}{R_2}$

TABLE 3.3 Typical Bridge Circuits Used in Instrumentation Circuits (*Continued*)

Bridge circuits	Balance relations
Owen bridge 	$L = C_1R_1R_2$ $R = \frac{C_1R_1}{C_2} - R_3$
Hay bridge 	$L = \frac{R_1R_3C}{1 + \omega^2C^2R_2^2}$ $R = \frac{\omega^2C^2R_1R_2R_3}{L + \omega^2C^2R_2^2}$

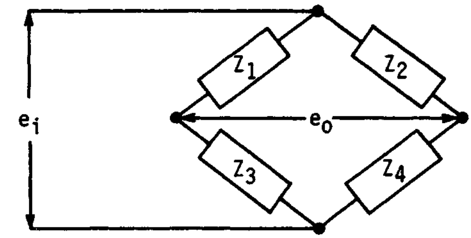


FIGURE 3.31 The bridge circuit.

where L = wire or foil length, A = wire or foil cross section, and ρ = electrical resistivity of the strain-gauge material. The strain-gauge sensitivity is the "*gauge factor*" GF given by the relation

$$\mathrm{GF} = \frac{\Delta R/R}{\Delta L/L} = 1 + 2\mu + \frac{\Delta\rho/\rho}{\Delta L/L} \tag{3.41}$$

where μ = Poisson's ratio for the strain-gauge material.

The strain gauge is often the sensing element in force transducers (load cells), pressure transducers, and accelerometers. The use of a strain gauge is illustrated in the following example.

Example 7. Figure 3.32 shows a rectangular cross section of a cantilever beam of width b and depth h with a bending load F applied at a distance L from where the strain is desired. A voltage-sensitive bridge circuit is used for excitation of 120-Ω, gauge factor 2.0 strain gauges.

The strain-gauge characteristics coupled with the bridge circuit and beam characteristics yield the following output voltage with respect to input load producing the strain:

$$e_o = Ke_i\left(\frac{\Delta R}{4R}\right) = \frac{Ke_i}{4}(\mathrm{GF})\,\epsilon = \frac{K}{4}e_i\,(\mathrm{GF})\,\frac{6FL}{Ebh^2} \tag{3.42}$$

K is the "bridge factor" in this equation and is a constant giving the magnification factor for using more than one active gauge in the bridge circuit. With two active gauges as shown, the bridge factor is 2 and the gauge arrangement gives complete compensation for temperature change of the beam. The temperature-induced strains are detected by the strain gauges but are effectively canceled in the bridge

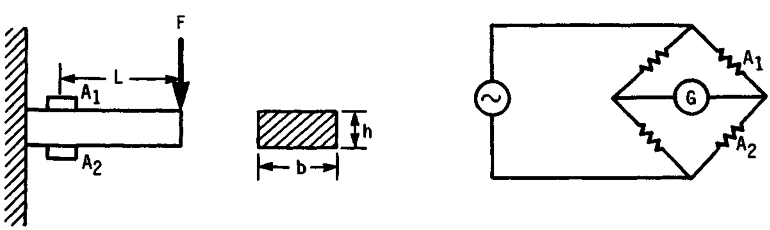


FIGURE 3.32 Cantilever beam with strain gauges.

circuit if the gauges are oriented to measure the same strain magnitude and if they have the same sensitivity; that is, matched transducers are used to cancel the "noise" signal caused by temperature change.

3.8 SOURCES OF ERROR IN MEASUREMENTS

The basic problem of every quantitative experiment is that of trying to identify the true value of the measured quantity. Philosophically, the measurement process is like viewing a deterministic event through a foggy window. Refinement of the measuring system to the ultimate should result in the measurement's being the true value. However, because errors occur in all measurements, one can never establish the true value of any quantity. Continued refinement of the methods used in any measurement will yield closer approximations to the true value, but there is always a limit beyond which refinement cannot be made. Furthermore, the fact that the measuring system draws energy from the source of the variable to be measured results in the measurement process's changing the characteristics of both the signal source and the measured variable. Thus some difference, however small, always occurs between the indicated value of the measured quantity and the original quantity to be measured.

3.8.1 Systematic Errors

Systematic errors are of consistent form. They result from conditions or procedures that cause a consistent error which is repeated every time the measurement is performed, such as faulty calibrations of the measuring system or changes in the measuring-system components due to factors such as aging.

Another form of systematic error can occur as a result of the observer. Parallax is an example of such an error. If the observer does not correctly align the indicating needle of the instrument, the reading may be consistently high or low depending on the individual observer. This type of error is difficult to detect because it is repeated with every measurement under identical conditions. However, one means of detecting systematic error is to measure something whose magnitude is accurately and independently known. For example, a check on a thermometer at one or more fixed points on the temperature scale will determine if the temperature scale on the thermometer is yielding a systematic error. Use of gauge blocks in checking micrometers and calipers is another example of checking a measuring system to eliminate systematic errors.

Calibration of the measuring system frequently, by use of accurately known input signals, can give an indication of the development of systematic errors in the measuring system. Measurement of the variable with two different measuring systems can often help detect the presence of a fixed error in the measurement system or measurement process.

3.8.2 Illegitimate Errors

Illegitimate errors are mistakes and should not exist. They may be eliminated by using care in the experimental procedure and by repetition in checking the measurement. Faulty data logging is an example of illegitimate error. This might occur by reading the wrong value from a scale, by writing down a wrong number, or by transposing the digits of a number. Another example of illegitimate error is that of using linear interpolation between scale divisions on a readout device when the scale is nonlinear. For example, some readout scales may be logarithmic, but it is quite common for observers to interpolate between these scale divisions in a linear fashion. Repeating the measurement and rechecking suspicious values can help eliminate such mistakes.

3.8.3 Random Errors

Random errors are accidental errors that occur in all measurements. They are characterized by their stochastic natures in both magnitude and time. One cannot determine their origin in the measurement process. These errors can only be estimated by statistical analysis. However, if both systematic and illegitimate errors can be eliminated, the uncertainty in the measurement due to the remaining random error can be estimated by statistical analysis of the data obtained in the experiment.

3.8.4 Loading Error

The loading error can be reduced in some measurement systems by means of a technique called *balancing*. A balance-type measurement is one where a reference signal is fed into the measurement system and a direct comparison between the reference and the measured signal is made. The reference signal is adjusted such that when its value is the same as that of the measured signal, the two signals balance one another and the output reading from the measurement system is zero. With the reference signal balancing out the measured signal, the net energy flow from the source at the balance condition is zero. Thus the balance method usually provides a more accurate measurement than the unbalance method. Examples of the balance type of measurement are the use of bridge circuits in strain-gauge measurement and the use of a voltage-balancing potentiometer with thermocouples when measuring temperatures. It should be noted that the balance type of measurement is usually difficult to achieve when dynamic or time-varying signals are being measured. The loading error is a type of fixed error in the measuring system and can be determined by appropriate calibration.

3.8.5 Noise-Measurement Systems

"Noise" in a measurement system is any output which is not generated by the input quantity to be measured ([3.4], [3.11], [3.12]). It must be remembered that all measuring systems are placed in an environment and interact in some way with that environment. Any interaction of a measuring system with the environment that is not related to the input quantity to be measured can result in an unwanted output (noise) of the measuring system. For noise to exist at a measurement-system output there must be both a source and a receiver of the noise. There must also be a coupling method between the source and the receiver of the noise.

The noise signal at the output of the measuring system can come from two general sources. One source is internally from the transducers in the measuring system, and the other source is from the environment of the transducers of the measuring system. Examples of internally generated signals are the thermal or Johnson noise

[3.4] created in a resister of an electric circuit. Another example is the shot noise [3.4] generated by tubes in electric circuits. External sources can cover a variety of possibilities, such as vibrations, electrical interference from the electromagnetic spectrum, and switching or discharge of storage elements causing transient signals to be induced in the power and signal lines of a measuring system. Any physical change in the environment can induce externally generated noise signals. These include temperature change, humidity change, pressure change, sound-level change, etc.

The noise signal can be *active* or *self-generating* in the sense that the noise is directly coupled to the measuring system without the aid of an auxiliary energy source. This might be thermoelectric or electromagnetic in nature. For example, a constantan strain gauge connected with copper lead wires could have a noise voltage generated by the thermocouple effect at the two junctions where the constantan gauge and copper lead wires are connected if the junctions are at different temperatures.

Noise effects that require the use of an auxiliary energy source to be carried into the measuring system are called *passive noise signals.* Examples of such noise signals are the temperature effects that occur with strain gauges and the strain effects that occur with resistance thermometers.

The effects of the noise on the output of the measuring system may be additive, multiplicative, or a combination of additive and multiplicative. If the noise level is an additive effect with no frequency content, the output signal due to the noise is called *zero shift.* This is a very common type of noise and can be easily detected by calibration. It is usually eliminated by a "bias" control on the measuring instrument. Noise levels that have a multiplicative effect on the system output usually affect the gain or sensitivity of the components of the measuring system. These effects can sometimes be detected by calibration.

At least four methods of handling noise are known ([3.4], [3.11], [3.12]). These include removal of the noise source, elimination of the noise by cancellation, minimization of the noise by division (filtering), and minimization of the noise by frequency-selective filtering. Removal of the noise source is not usually possible, and one must generally resort to the other techniques. However, if the noise source can be eliminated, this will be the most effective method of preventing the noise problem.

When the effects which create the noise are consistent to the extent that one can expect two matched transducers to detect identical noise signals at the same instant of time in the same environment, it is possible to arrange for cancellation of these signals by subtraction. For example, the weight of a balance-scale pan may be subtracted (or balanced out) by placing an equal weight on the other side of the balance scale. Another example is the temperature-induced noise in strain gauges illustrated in Example 7. The temperature-induced resistance change in a strain gauge can be canceled out by placing two identical strain gauges (matched transducers) in the same thermal environment and by using proper placement of the gauges in a bridge circuit to provide subtraction of the noise signals. This is called *temperature compensation.* Another example is the noise-canceling microphone, in which two sensing elements are placed opposite one another. Voice input is supplied to only one element while external noise is sensed by both elements and is effectively balanced out or canceled. This technique is often used in aircraft applications. The use of this technique for noise elimination depends on being able to have two identical sensing elements which detect and respond to the noise signal to which they are exposed in exactly the same way and at the same time. Such detector elements are called *matched transducers.*

When the effects which create the noise level are not consistent, one cannot expect that two noise sources under identical environmental conditions will emit the same noise at every instance of time. In this case, noise is minimized by division so that only a small fraction of the original noise propagates through the system. For example, contact-resistance phenomena in switching gear and slip rings cause this type of noise. Electric circuitry in the measuring system is designed so that these resistance changes will have a minimal effect on the output reading. Electromagnetic radiation can also cause this type of noise input. Appropriate shielding of lead wires and circuits is necessary and is commonly used to minimize this type of noise.

Frequency-selecting filtering can be used if the noise and the desired signal can be made to exist at different frequencies [3.4]. When this is the case, a simple filter may then be used to minimize the noise signal. If the signal and the noise exist in the same frequency range, one must resort to a technique of modulation where the signal frequency is moved to a frequency range sufficiently separated from the noise frequency that the noise frequency can be effectively filtered from the signal frequency. This technique of frequency-selective filtering can be used to minimize only active noise in a passive transducer, since the carrier wave of the auxiliary energy source is used via modulation to shift the signal frequency upward and separate it from the noise frequency. If the noise is passive noise (depends on the auxiliary source of energy) in the transducer, the signal and noise would both be modulated upward to the same frequency band, and separation could not be achieved. References [3.11] and [3.12] give specific details on how noise can be eliminated or minimized in a given measurement situation.

3.8.6 Precision and Accuracy

Accuracy is the difference between a measured variable and the true value of the measured variable. Normally, one is not able to determine accuracy. *Precision* is the difference between a measured variable and the best estimate (as obtained from the measured variable) of the true value of the measured variable. Thus precision is a measure of repeatability. It should also be noted that one may have excellent precision in a measurement but very poor accuracy. Calibration of a measuring system is essential in determining its accuracy. Precision is specified by quantities called *precision indices* (denoted by W_x) that are calculated from the random errors of a set of measurements.

When a variable is measured, it is just as important to state the precision of the measurement as it is to state the most representative value of the quantity. Thus we desire W_x to be specified for every measured variable. The confidence or probability for obtaining the range $\pm W_x$ is generally specified directly or else is implied by the particular type of precision index being used.

3.9 ANALYSIS OF DATA

The basic problem in every quantitative experiment is that of obtaining the unbiased estimate $\hat{\mu}$ of the true value μ of a quantity as well as an unbiased estimate $\hat{W}$ of the dispersion or uncertainty in the measured variable. Data sets or samples typically display the two very important characteristics of central tendency (or most representative value) and dispersion (or scatter). Other characteristics, such as skewness and flatness (or peakness), may also be of importance but are not considered in the items that follow.

If we were given a list of measured values of the same variable, the question is raised as to what value shall be taken as being nearest to the true value. In order to determine what relation the measured value has to the true value, we must be able to specify in any experiment the unbiased estimate $\hat{\mu}$ of the true value of a measurement and its uncertainty (or precision) interval W based on a given confidence level (or probability of occurrence).

3.9.1 Unbiased Sampling

An *unbiased estimator* exists if the mean of its distribution is the same as the quantity being estimated [3.13]. Thus for sample mean $\bar{x}$ to be an unbiased estimator of population mean μ, the mean of the distribution of sample means $\bar{\bar{x}}$ must be equal to the population mean.

It can be shown that the unbiased estimator $\hat{\mu}$ for the population mean μ is the sample mean $\bar{x}$. In this section the measure of dispersion is selected to be the standard deviation σ or its square σ^2, called the *variance.* Determination of the unbiased estimator of the standard deviation or variance depends on the type of sampling method used.

Figure 3.33 illustrates that different samples from a population yield slightly different estimates of population mean and variance. However, if the data from the individual samples are combined, even better estimates of population mean and variance can be achieved. The mean of the sample means $\bar{\bar{x}}$ is a better estimate of μ than any of the individual sample means $\bar{x}$. Also, the dispersion of the distribution of $\bar{x}$ values is much less than the dispersion of items within an individual sample, as indicated by the central limit theorem.

The *central limit theorem* yields the result that if one obtains random samples of size n from a large population of mean μ and variance σ^2, the distribution of sample means approaches gaussian as n becomes large with a mean μ and a variance σ^2/n. This is valid regardless of the nature of the distribution of the population from which the sample values were obtained.

A *random sample* is a sample collected such that every member of the population from which one is sampling has an equal probability of selection in every trial. This may be done with or without replacement. For a sample of size n, the probability of selection from the population for each member of the sample is $1/n$.

Unbiased estimates for determining population mean, population variance, and variance of the sample means depend on the type of sampling procedure used. Unbiased estimates of population mean μ, population variance σ^2, and variance of the mean are listed in Table 3.4 for sampling both with and without replacement. Note from the relations in Table 3.4 that sampling without replacement from an extremely large population is essentially equivalent to sampling with replacement (random sampling) since $(N-1)/N$ and $(N-n)/(N-1)$ approach unity.

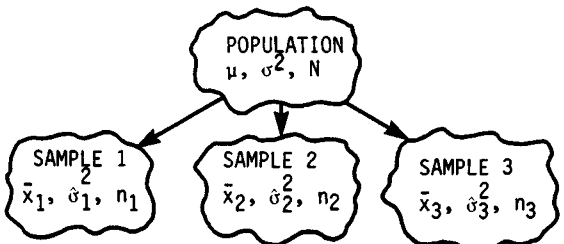


FIGURE 3.33 Sampling from a population.

TABLE 3.4 Unbiased Estimates of Mean, Variance, and Variance of the Mean†

Statistic	Sampling with replacement	Sampling without replacement
Mean	$\hat{\mu} = \frac{\Sigma x_i}{n} = \bar{x}$	$\hat{\mu} = \frac{\Sigma x_i}{n} = \bar{x}$
Variance	$\hat{\sigma}^2 = \frac{\Sigma(x_i - \bar{x})^2}{n-1} = S^2\left(\frac{n}{n-1}\right)$	$\hat{\sigma}^2 = S^2\left(\frac{n}{n-1}\right)\left(\frac{N-1}{N}\right)$
Variance of the mean	$\sigma_{\bar{x}}^2 = \frac{\hat{\sigma}^2}{n}$	$\hat{\sigma}_{\bar{x}}^2 = \frac{\hat{\sigma}^2}{n}\left(\frac{N-n}{N-1}\right)$

†S^2 = sample variance = $\Sigma(x_i - \bar{x})^2/n$ and $n/(n-1)$ = Bessel's correction.

Example 8. The following data set is obtained by sampling from a population of 15 items:

x_i	$(x_i - \bar{x})$	$(x_i - \bar{x})^2$
65.0	−15.3	234.09
73.1	−7.2	51.84
83.0	+2.7	7.29
100.1	+19.8	392.04
$\Sigma = 321.2$	00.0	685.26

1. Determine the mean and standard deviation of the *original data.*

$$n = 4 \qquad \bar{x} = \frac{321.2}{4} = 80.3$$

$$s = \sqrt{\frac{\Sigma(x_1 - \bar{x})^2}{n}} = \sqrt{\frac{685.26}{4}} = \sqrt{171.3} = \pm 13.09$$

2. What are the estimates of *population mean* and *variance* if the sampling occurred *with* replacement?

$$\hat{\mu} = \bar{x} = 80.3$$

$$\hat{\sigma}^2 = s^2 \frac{n}{n-1} = 171.3 \frac{4}{3} = 228.4$$

3. What are the estimates of population mean and variance if the sampling occurred *without* replacement?

$$\hat{\mu} = \bar{x} = 80.3$$

$$\hat{\sigma}^2 = s^2 \frac{n}{n-1} \frac{N-1}{N} = 171.3 \frac{4}{3} \frac{14}{15} = 213.2$$

4. What is the variance associated with the distribution of $\bar{x}$ values if sampling occurred *without* replacement?

$$\hat{\sigma}_{\bar{x}}^2 = \frac{N-n}{N-1}\frac{\hat{\sigma}_x^2}{n} = \frac{(15-4)213.17}{(15-1)4} = 41.87$$

3.9.2 Uncertainty Interval

When several observations of a variable have been obtained to form a data set (multisample data), the best estimates of the most representative value (mean) and dispersion (standard deviation) are obtained from the formulas in Table 3.4. When only a single measurement exists (or when the data are taken so that they are similar to a single measurement), the standard deviation cannot be determined and the data are said to be "single-sample" data. Under these conditions, the only estimate of the true value is the single measurement, and the uncertainty interval must be estimated by the observer. Kline and McClintock [3.14] address this problem in detail. It is recommended that the precision index be estimated as the maximum error and that it correspond approximately to the 99 percent confidence level associated with multisample data.

Once the unbiased estimates of mean and variance are determined from the sample data, the *uncertainty interval* can be expressed as

$$\mu = \hat{\mu} \pm \hat{W} = \hat{\mu} \pm k(\nu, \gamma)\hat{\sigma} \tag{3.43}$$

where $\hat{\mu}$ = the most representative value of the measured data and $\hat{W}$ = the uncertainty interval or precision index associated with the estimate of μ. The magnitude of the precision index or uncertainty interval depends on confidence level γ (or probability chosen), sample size n, and type of probability distribution governing the distribution of measured items.

The uncertainty interval $\hat{W}$ can be replaced by $k\hat{\sigma}$, where $\hat{\sigma}$ is the standard deviation (measure of dispersion) of the population as estimated from the sample and k is a constant that depends on the probability distribution, the confidence level γ, and the sample size n. For example, with a gaussian distribution, the 95 percent confidence limits are $\hat{W} = 1.96\hat{\sigma}$, where $k = 1.96$ and in this case is independent of n. For a t distribution, $k = 2.78, 2.06$, and 1.96 for sample sizes of 5, 25, and ∞, respectively, at the 95 percent confidence level. The t distribution becomes the gaussian distribution as $n \rightarrow \infty$. The uncertainty interval $\hat{W}$ in Eq. (3.43) assumes a set of measured values with only random error present. Furthermore, the set of measured values is assumed to have unbounded significant digits and to have been obtained with a measuring system having infinite resolution. When *finite resolution* exists and *truncation of significant digits* occurs, the uncertainty interval will be larger than that predicted by consideration of only the random error [3.15]. The uncertainty interval can never be less than the resolution limits or truncation limits of the measured values. If $\{s_n\}$ is the theoretically possible set of measurements of unbounded resolution and $\{x_n\}$ is the actual set of measurements expressed to m significant details from a measuring system of finite resolution R, the quantity $s_i - x_i = \pm e_i$ is the resolution or truncation deficiency caused by the measurement process. The unbiased estimates of mean and variance are

$$\hat{\mu} = \frac{\Sigma s_i}{n} = \bar{s} \qquad \hat{\sigma}^2 = \frac{\Sigma(s_i - \bar{s})^2}{n-1} \tag{3.44}$$

Noting that the experimenter has the set $\{x_n\}$ rather than $\{s_n\}$, the mean and variance become

$$\hat{\mu} = \frac{\Sigma x_i}{n} \pm \frac{\Sigma e_i}{n} = \bar{x} \pm \frac{\Sigma e_i}{n} \qquad \hat{\sigma}^2 = \frac{\Sigma(x_i - \bar{x})^2}{n-1} \tag{3.45}$$

Thus the truncation or resolution has no effect on the estimate of variance but does effect the estimate of the mean.

The truncation errors e_i are not necessarily distributed randomly and may all be of the same sign. Thus $\bar{x}$ can be biased as much as $\Sigma e_i/n = \bar{e}$ high or low from the unbiased estimate of the value of μ, so that $\hat{\mu} = \bar{x} \pm \bar{e}$.

If e_i is a random variable, such as when observing a variable with a measuring system of finite resolution, the values of e_i may be plus or minus, but their upper bound is R (the resolution of the measurement). Thus the resolution error is no larger than R, and $\mu = \bar{x} \pm R$.

If the truncation is never more than that dictated by the resolution limits R of the measuring system, the uncertainty in $\bar{x}$ as a measure of the most representative value of μ is never larger than R plus the uncertainty due to the random error. Thus $\hat{\mu} = \bar{x} \pm (\hat{W} + R)$. It should be emphasized that the uncertainty interval can never be less than the resolution bounds of the measurement no matter how small the random error might be. The resolution bounds cannot be reduced without changing the measurement system.

When x_i is observed to m significant digits, the uncertainty (except for random error) is never more than $\pm$ $(5/10^m)$, and the bounds on s_i are equal to $x_i \pm (5/10^m)$, so that

$$x_i - \frac{5}{10^m} < s_i < x_i + \frac{5}{10^m} \tag{3.46}$$

The relation for $\hat{\mu}$ for m significant digits is then

$$\hat{\mu} = \bar{x} \pm \frac{\Sigma e_i}{n} = \bar{x} \pm \frac{\Sigma\, 5/10^m}{n} = \bar{x} \pm \frac{5}{10^m} \tag{3.47}$$

When the uncertainty due to significant digits is combined with the resolution limits and random error, the uncertainty interval on μ becomes

$$\mu = \hat{\mu} \pm \left(\hat{W} + R + \frac{5}{10^m}\right) \tag{3.48}$$

This illustrates that the number of significant digits of a measurement should be carefully chosen in relation to the resolution limits of the measuring system so that $5/10^m$ has about the same magnitude as R. Additional significant digits would imply more accuracy to the measurement than would actually exist based on the resolving ability of the measuring system.

3.9.3 Amount of Data to Take

Exactly what data to take and how much data to take are two important questions to be answered in any experiment. Assuming that the correct variables have been measured, the amount of data to obtain can be determined by using the relation

$$\mu = \hat{\mu} \pm \left(\hat{W}_x + R + \frac{5}{10^m}\right) \tag{3.49}$$

where it is presumed that several samples may exist for estimation of μ. This equation can be rewritten such that

$$\mu = \bar{\bar{x}} \pm \left[k(\nu, \alpha) \frac{\hat{\sigma}}{\sqrt{n}} + R + \frac{5}{10^m} \right] \tag{3.50}$$

If one wishes to know the value of n to achieve the difference in $\mu - \bar{x}$ within a stated percent of μ, the relation can be solved for n to yield

$$n^2 = \frac{k(\nu, \gamma)\, \hat{\sigma}}{(\text{percent}/100)\hat{\mu} - R - (5/10^m)} \tag{3.51}$$

This equation can yield valid values of n only once estimates of $\hat{\mu}$, $\hat{\sigma}$, k, R, and m are available. This means that the most correct value of n can be obtained only once the measurement system and data-taking procedure have been specified so that R and m are known. Furthermore, either a preliminary experiment or a portion of the actual experiment should be performed to obtain good estimates of $\hat{\mu}$ and $\hat{\sigma}$. Because k depends on the type of data distribution, the sample size n yields an iterative reduction. Thus the most valid estimates of the amount of data to take can be obtained only after the experiment has begun. This requires that estimates of the mean and the standard deviation be obtained by performing part of the experiment. However, the equation can be quite useful for prediction purposes if one wishes to estimate values of $\hat{\mu}$, $\hat{\sigma}$, k, R, and m. This is especially important in experiments where the cost of a single run may be relatively high.

Example 9. The life (mileage) for a certain type of automotive tire is known to follow a gaussian (normal) distribution function. The mean and standard deviation of the mileage for these tires are estimated to be 84 000 and 2100 mi, respectively, from a sample of nine tires. Determine the 90 percent confidence limits for the means of all such tires manufactured by the company if the resolution of these measurements is 5 mi. On the basis of the sample, how much data (i.e., what is the sample size?) are required to establish the life of this type of tire to within ±1 percent with 90 percent confidence and a resolution of 5 mi?

Solution

$$\begin{aligned}
\mu &= \mu \pm (t\sigma_{\bar{x}} + R) \\
\sigma_x &= 2230 \text{ mi} \\
\hat{\sigma}_{\bar{x}} &= \frac{\sigma_x}{\sqrt{n}} = \sqrt{\frac{\Sigma(x_i - \bar{x})^2}{n(n-1)}} = \pm 743 \text{ mi} \\
t &= t(\nu, \gamma) = t(8, 0.90) = 1.860^{\dagger} \\
\mu &= 84\,000 \pm [1.860(743) + 5] \\
&= 84\,000 \pm 1387 \text{ mi} \\
\mu &= \hat{\mu} \pm (t\hat{\sigma}_x + R) \\
\mu - \hat{\mu} &= 0.01\bar{x} = t\frac{\hat{\sigma}_x}{\sqrt{n}} + R \\
\frac{t}{\sqrt{n}} &= \frac{0.01\bar{x} - R}{\hat{\sigma}_x} = \frac{835}{2230} = 0.374
\end{aligned}$$

† From Ref. [3.7], Table A-8.

Use of Table A-8 in Ref. [3.7] yields the final result of $n = 122$.

3.10 CONFIDENCE LIMITS

A "confidence" limit or uncertainty interval is associated with a probability. The area under a probability-density curve between any two limits gives the value of the probability or confidence that any item sampled at random from the population will have a value between the two limits chosen. For example, the area under the gaussian (or normal) probability-density function $p(x)$ between values of x_1 and x_2 is given by

$$\int_{x_1}^{x_2} p(x)\,dx = \int_{x_1}^{x_2} \frac{1}{\sigma\sqrt{2\pi}} \exp\left[-\frac{(x-\mu)^2}{2\sigma^2}\right] dx = \gamma \tag{3.52}$$

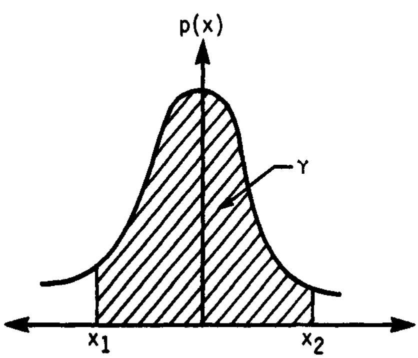


FIGURE 3.34 Confidence limits on the gaussian distribution.

and represents (as shown in Fig. 3.34) the probability that any one item selected at random from the population of values will have a magnitude between x_1 and x_2. In this figure, x_1 and x_2 are called *precision indices.* The symbol W_x is used to denote the values of x to be taken as a precision index or uncertainty interval. For the gaussian distribution, a value of the precision indices of $\pm\sigma$ (one standard deviation) yields a probability or confidence of 68.3 percent. Also, $W_x = \pm 1.966$ are the 95 percent confidence limits and $W_x = \pm 2.586$ are the 99 percent confidence limits for the population of items following the gaussian distribution. The confidence limits and associated probability are illustrated in Fig. 3.34. This information is often represented by the following probabilistic statement:

$$p(x_1 < x < x_2) = \gamma = 1 - \alpha \tag{3.53}$$

where γ = the probability of the value of x from an observation to be between the values of x_1 and x_2. The value γ is known as the *confidence level,* whereas the value α is known as the *significance level.* The meaning of the one-sided probabilistic statements

$$p(x < x_3) = \gamma \qquad \text{and} \qquad p(x > x_3) = \alpha = 1 - \gamma \tag{3.54}$$

is illustrated in Fig. 3.35.

Example 10. Certain strain gauges are manufactured with a resistance specification of $120 \pm 0.5\ \Omega$. All gauges not meeting this specification are rejected. If all such strain gauges manufactured follow a gaussian probability function, estimate the standard deviation of the manufacturing process if 2 percent are typically rejected (refer to Fig. 3.36 and Table A-4 of Ref. [3.7]).

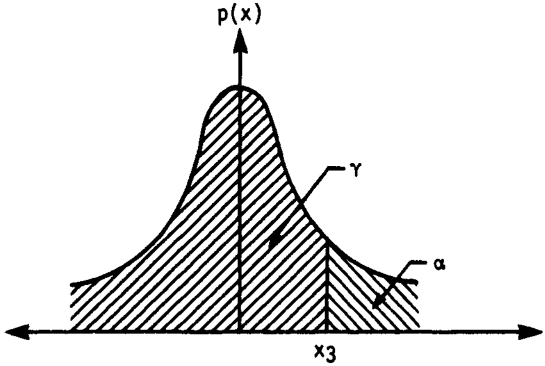


FIGURE 3.35 Single-sided probabilistic statements.

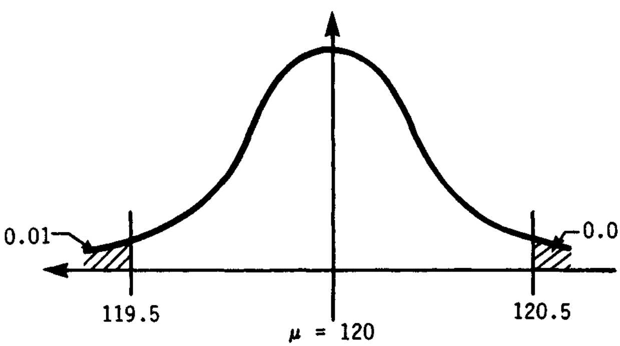


FIGURE 3.36 Gaussian distribution for Example 10.

Solution

$$p(119.5 < R < 120.5) = 0.98$$

$$0.01 = \int_{z_1}^{\infty} f(z)\, dz = 0.500 - \int_0^{z_1} f(z)\, dz$$

$$\int_0^{z_1} f(z)\, dz = 0.490 \qquad z_1 = 2.323$$

$$z = \frac{x - \mu}{\sigma} = \frac{120.5 - 120.0}{\sigma} = 2.323$$

$$\sigma = \frac{0.5}{2.323} = 0.215\ \Omega$$

3.10.1 Confidence Limits on Means

The confidence limits on the establishment of the mean μ of a population from sample data are established [3.13] by the probabilistic statement

$$P\left[-t(\nu,\gamma) < \frac{\bar{x}-\mu}{\sigma/\sqrt{n}} < t(\nu,\gamma)\right] = \gamma \tag{3.55}$$

where $t(\nu, \gamma)$ is the well-known t statistic. Rearrangement of Eq. (3.55) yields the probabilistic statement showing how to establish confidence limits on μ:

$$P\left[\left(\bar{x} - t\frac{\sigma}{\sqrt{n}}\right) < \mu < \left(\bar{x} + \frac{t\sigma}{\sqrt{n}}\right)\right] = \gamma \tag{3.56}$$

The effects of measurement resolution and significant digits can be included in the expression to yield Eq. (3.57). This equation shows that even if the random error is zero, the uncertainty on μ cannot be less than the resolution and truncation uncertainties:

$$P\left[\left(\bar{x} - \frac{t\sigma}{\sqrt{n}} - R - \frac{5}{10^m}\right) < \mu < \left(\bar{x} + \frac{t\sigma}{\sqrt{n}} + R + \frac{5}{10^m}\right)\right] = \gamma \tag{3.57}$$

Pooling or Combining Data. If one has several samples (perhaps obtained by different instrument systems) for estimating a population mean μ and variance σ^2, these data can be formally combined to obtain better estimates of these entities. If each data set comes from a normal population of mean μ and variance σ^2, the maximum-likelihood technique can be used to show how the means and variances of the samples are combined to provide the better estimates:

$$\hat{\mu}_c = \frac{\Sigma n_j \bar{x}_j/\hat{\sigma}_j^2}{\Sigma n_j/\hat{\sigma}_j^2} = \frac{n_1\bar{x}_1/\hat{\sigma}_1^2 + n_2\bar{x}_2/\hat{\sigma}_2^2 + \cdots}{n_1/\hat{\sigma}_1^2 + n_2/\hat{\sigma}_2^2 + \cdots} \tag{3.58}$$

$$\hat{\sigma}_c^2 = \frac{\Sigma n_j}{\Sigma n_j/\hat{\sigma}_j^2} = \frac{n_1 + n_2 + \cdots}{n_1/\hat{\sigma}_1^2 + n_2/\hat{\sigma}_2^2 + \cdots} \tag{3.59}$$

The reason for pooling data is to obtain more precise estimates of μ and σ^2 than any sample alone can provide. The intuitive thought, to use the more precise data and to discard the less precise data, *is not* the appropriate thing to do. Instead, the uncertainty intervals on the estimate of population mean μ and variance σ^2 are considerably reduced by pooling the data.

Example 11. The data in the sample table below represent shear strengths in pounds per square inch (psi) of an adhesive. To market this adhesive, what should its guaranteed shear strength be with a confidence level of 99.5 percent?

Sample 1	Sample 2	Sample 3	Sample 4
3566	3180	3470	3520
3630	3080	3620	3960
3800	3400	3466	4040
3880	3820	3460	3600
3840	3480	3760	3899

Solution

$$\hat{\mu}_1 = 3742.2 \qquad \hat{\mu}_2 = 3392.0 \qquad \hat{\mu}_3 = 3555.2 \qquad \hat{\mu}_4 = 3803.8$$

$$\hat{\sigma}_1^2 = 18\,880 \qquad \hat{\sigma}_2^2 = 83\,334 \qquad \hat{\sigma}_3^2 = 17\,612 \qquad \hat{\sigma}_4^2 = 70\,567$$

$$\hat{\mu}_c = 3639.7 \text{ psi} \qquad \hat{\sigma}_c^2 = 29\,431 \text{ (psi)}^2 \qquad \hat{\sigma}_c = 171.3 \text{ psi}$$

$$\begin{aligned} \mu &= \hat{\mu} \pm \left(t\frac{\hat{\sigma}}{\sqrt{n}} + \frac{5}{10^m} + R \right) = 3639.7 \pm 2.921\,\frac{(171.3)}{\sqrt{20}} \\ &= 3640 \pm 108 \end{aligned}$$

where $t = t(\nu, \gamma) = t(16, 0.99) = 2.921$ from Ref. [3.7], Table A-8. The strength should be guaranteed to $3640 - 108 = 3532$ psi.

3.10.2 Confidence Limits on Variance

To establish the confidence limits on variance, the chi-square statistic $\chi^2 = [\Sigma(x_i - \bar{x})^2]/\sigma^2$ is used [3.13], and the probabilistic statement becomes

$$P\left[\chi_L^2 < \frac{\Sigma(x_i - \bar{x})^2}{\sigma^2} < \chi_R^2\right] = \gamma \tag{3.60}$$

Rearranging yields the probabilistic statement showing how to determine confidence limits on variance:

$$P\left[\frac{\Sigma(x_i - \bar{x})^2}{\chi_R^2} < \sigma^2 < \frac{\Sigma(x_i - \bar{x})^2}{\chi_L^2}\right] = \gamma \tag{3.61}$$

Example 12. The standard deviation of the lifetimes of a sample of 200 high-pressure seals is 100 h. What are the 95 percent confidence limits on the standard deviation of all such seals? (See Ref. [3.7], Table A-7, for $\nu > 30$.)

Solution

$$\sqrt{2\chi^2} - \sqrt{2\nu - 1} \sim N(0, 1)$$

$$\therefore\ \sqrt{2\chi^2} - \sqrt{2\nu - 1} = z \qquad \text{or} \qquad \chi^2 = \tfrac{1}{2}(z + \sqrt{2\nu - 1})^2$$

$$\begin{aligned} \chi_R^2 &= \chi^2(0.025, 199) \\ &= \tfrac{1}{2}(+z_{0.025} + \sqrt{2(199) - 1})^2 \\ &= \tfrac{1}{2}(+1.96 + 19.92)^2 = 239 \end{aligned}$$

$$\chi_R = \sqrt{239} = 15.4$$

$$\begin{aligned} \chi_L^2 &= \tfrac{1}{2}(+z_{0.975} + \sqrt{2(199) - 1})^2 \\ &= \tfrac{1}{2}(-1.96 + 19.92)^2 = 161 \end{aligned}$$

$$\chi_L = \sqrt{161} = 12.7$$

$$P\left(\frac{\hat{\sigma}\sqrt{n-1}}{\chi_R} < \sigma < \frac{\hat{\sigma}\sqrt{n-1}}{\chi_L}\right) = \gamma$$

$$\therefore\ P(91.2 < \sigma < 111.3) = 0.95$$

3.11 PROPAGATION OF ERROR OR UNCERTAINTY

In many cases the desired quantity and its uncertainty cannot be measured directly but must be calculated from the data of two or more measured variables. This is represented mathematically by

$$R = R(x_1, x_2, x_3, \ldots, x_n) \tag{3.67}$$

where the x's = measured variables and R = the dependent or calculated quantity. To determine the most representative value and uncertainty of the calculated quantity, the following equations can be used [3.15]:

$$\mu_R = R(\mu_{x_1}, \mu_{x_2}, \mu_{x_3}, \ldots, \mu_{x_n}) + \frac{1}{2}\Sigma\left(\frac{\partial^2 R}{\partial x_i^2}\right)\sigma_{x_i}^2 \tag{3.68}$$

$$\sigma_R^2 = \Sigma\left(\frac{\partial R}{\partial x_i}\right)_\mu^2 \sigma_{x_i}^2 + \frac{1}{2}\Sigma\left(\frac{\partial^2 R}{\partial x_i^2}\right)_\mu^2 \sigma_{x_i}^4 + \cdots \tag{3.69}$$

Example 13. If $r = x^n$, then μ_R and σ_R^2 become

$$\mu_R = (\mu_x)^n\left[1 + \tfrac{1}{2}(n)(n-1)\left(\frac{\sigma_x}{\mu_x}\right)^2\right] \tag{3.70}$$

$$\sigma_R^2 = n^2(\mu_x)^n\left[1 + \tfrac{1}{2}(n-1)^2\left(\frac{\sigma_x}{\mu_x}\right)^2\right] \tag{3.71}$$

The role of the coefficient of variation σ_x/μ_x should be noted. If σ_x/μ_x is small, the first terms in each of Eqs. (3.68) and (3.69) are all that need be evaluated to yield the desired results. The results for several other functions are given by Mischke [3.15].

The variances in Eqs. (3.68) and (3.69) can be replaced with confidence limits according to the equation

$$W_x = k\sigma_x \tag{3.72}$$

where k = a constant depending on the type of data distribution. The propagation-of-variance equation becomes the following propagation-of-uncertainty formula:

$$W_R^2 = \Sigma\left(\frac{\partial R}{\partial x_i}\right)_\mu^2 W_{x_i}^2 \tag{3.73}$$

Use of this equation requires that all the x_i's be independently measured, that the k values be the same for each x_i distribution, and that all W_{x_i}'s represent the same level of uncertainty or confidence.

Example 14. The strain and its uncertainty a distance L from the load F on the top surface of the cantilever beam shown in Fig. 3.32 is to be determined subject to the following measured data:

$$b = 0.500 \pm 0.001 \text{ in}$$

$$h = 0.250 \pm 0.001 \text{ in}$$

$$L = 20.00 \pm 0.01 \text{ in}$$

$$F = 10.00 \pm 0.01 \text{ lbf}$$

$$E = (30.0 \pm 0.01) \times 10^6 \text{ psi}$$

Solution. The strain is determined by $\epsilon = 6FL/(Ebh^2) = 1280 \times 10^{-6}$ in/in, or $\epsilon =$ 1280 microstrain. The uncertainty in the strain is determined by

$$W_\epsilon^2 = \left(\frac{\partial \epsilon}{\partial F}\right)_\mu^2 W_F^2 + \left(\frac{\partial \epsilon}{\partial L}\right)_\mu^2 W_L^2 + \left(\frac{\partial \epsilon}{\partial E}\right)_\mu^2 W_E^2 + \left(\frac{\partial \epsilon}{\partial b}\right)_\mu^2 W_b^2 + \left(\frac{\partial E}{\partial h}\right)_\mu^2 W_h^2 \quad (3.74)$$

This simplifies to the following equation when all the coefficients of variation are small:

$$\left(\frac{W_\epsilon}{\bar{\epsilon}}\right)^2 = \left(\frac{W_F}{\bar{F}}\right)^2 + \left(\frac{W_L}{\bar{L}}\right)^2 + \left(\frac{W_E}{\bar{E}}\right)^2 + \left(\frac{W_b}{\bar{b}}\right)^2 + 4\left(\frac{W_h}{\bar{h}}\right)^2 \quad (3.75)$$

Substitution of the data in Eq. (3.75) yields $W_\epsilon = \pm 11$ microstrain. When the second terms in Eqs. (3.68) and (3.69) are used, the results are $\epsilon = 1280.1$ microstrain and $W_\epsilon = \pm 11.3$ microstrain, which illustrates that when the coefficients of variation are small, the additional terms do not contribute significantly to ϵ and W_ϵ.

A simplified form of the preceding propagation-of-uncertainty equation results if the function R has the special form

$$R = (X_1^a X_2^b \cdots X_n^m)K \quad (3.76)$$

where K = any constant and the exponents a, b, and m may be positive or negative, integer or noninteger. Substitution of Eq. (3.76) into Eq. (3.73) yields

$$\left(\frac{W_R}{\bar{R}}\right)^2 = a^2\left(\frac{W_{x_1}}{\bar{x}_1}\right)^2 + b^2\left(\frac{W_{x_2}}{\bar{x}_2}\right)^2 + \cdots + m^2\left(\frac{W_{x_n}}{\bar{x}_n}\right)^2 \quad (3.77)$$

The equation for uncertainty in the calculated quantity allows one to see the effect of the exponents $a, b, \ldots, m$ in propagation of the measured variable uncertainties to the uncertainty in the calculated quantity. For example, a squared variable x_i^2 has four times the effect on the propagation of uncertainty in the result that it would have if it had a unity exponent. It should also be noted that all terms are dimensionless in Eq. (3.77) and that the ratios $W_{x_i}/\bar{x}_i$, called *relative error,* are proportional to coefficients of variation by the factor k, since $W_x = k\sigma_x$.

If the experimental data are "single sample," the values of W_{x_i} must be estimated by the experimentor, as indicated by Kline and McClintock [3.14]. It is suggested that with single-sample data the W_{x_i} be estimated as the maximum error (which corresponds approximately to the 99 percent confidence level).

The propagation-of-uncertainty equation is extremely valuable in planning experiments. If one desires a certain precision on the calculated result R, the precision of the measured variables can be determined from this equation. Because instrument precision is directly related to its cost, one also has a method of estimating costs of a proposed measuring system by use of the propagation equation.

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